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A simple receding horizon control for state delayed systems and its stability criterion

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Abstract

In this paper, a simple receding horizon (or model predictive) control for state delayed systems is presented and its solution is given in a closed form by a reduction method. While the control for a time-delay system is usually complex, the proposed controller is simple to construct and therefore can be simply implemented in real applications. To check the closed-loop stability of the proposed controller, a sufficient condition is provided by linear matrix inequalities. In addition, a numerical algorithm is presented for computing the eigenvalues of systems with distributed time delays, which can be used as a necessary and sufficient condition to check closed-loop stability. It is shown by simulation that this simple control can be a stabilizing control for time-delay systems.

Keywords: Receding horizon control; Time-delay system; Linear matrix inequalities

1. Introduction

It is generally recognized that delays are natural components of many dynamic processes. Therefore, over many decades, there has been considerable research to produce stabilizing controls for time-delay systems.

First, infinite-time (steady state) optimal controls minimizing quadratic performance criteria can be used for stabilizing control designs [1]. The optimal control produced for state delayed systems consists of feedback of the current states, and the integral of past states. However, the feedback gain matrices are very difficult to compute because they are associated with several coupled partial differential equations with two-point boundary value conditions [1–3]. Only approximate solutions have been produced using numerical techniques [3,4].

Secondly, there have been other attempts to obtain stabilizing controls by using only sufficient conditions, such as Lyapunov methods [5–12]. The conditions often give only memoryless controls, and general feedback controls with distributed delays are difficult to derive.

Meanwhile, it is well known that in ordinary systems, receding horizon (or model predictive) controls provide stabilizing controls [13–20]. In particular, there is an extremely simple way to stabilize linear systems [13]. For time-delay systems, general stabilizing receding horizon controls are currently unknown. However, in this paper, a simple but reasonably general receding horizon control, similar to that for ordinary systems, will be suggested for time-delay systems. We introduce a reduction technique so that the optimal problem for state-delay systems can be transformed to an optimal problem for delay-free ordinary systems. Unlike the corresponding ordinary systems, the stability of this simple control is yet to be proved, although it is conjectured that this control can stabilize a broad class of time-delay systems.

In order to check stability of the proposed control, we provide two different schemes. First, we present a sufficient stability condition in terms of linear matrix inequalities (LMIs), which can be easily checked using convex optimization algorithms [21]. It is believed that the closed-loop stability can be checked only by this sufficient condition in most cases. However, the generated

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control may still be stable even if the stability condition is not met. We provide a numerical algorithm to compute the eigenvalues of time-delay systems, which can be used as a necessary and sufficient condition to check the closed-loop stability of the proposed controller. In [22], a numerical algorithm was proposed to compute the eigenvalues of time-delay systems, which was not extended to the case of more general time-delay systems with distributed delay terms. The algorithm proposed in this paper is applied to general time-delay systems with distributed delays. It will be shown through simulations that state-delay systems can be stabilized by the suggested method.

The organization of the paper is as follows. In Section 2, a simple receding horizon control for state delayed systems is derived. In Section 3, a sufficient condition in terms of LMIs for the closed-loop stability is given. In Section 4, a numerical algorithm is presented for computing eigenvalues of general distributed delay systems and applied to check the stability of the proposed controller. Numerical examples are given in Section 5, and finally, Section 6 presents the conclusions.

2. A simple receding horizon control for state delayed systems

Let us consider a linear system with a delayed state

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + B_0u(t) \tag{1}$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [t_0 - h, t_0]$$

where $x(t) \in R^n$, $u(t) \in R^m$, A_0, $A_1 \in R^{n\times n}$, $B_0 \in R^{n\times m}$, $h>0$ is the delay, and $\phi(t)$ is a continuous function. First, we find the optimal control which minimizes a cost function defined by

$$J = \int_{t_0}^{t_1} u^T(t)Ru(t)\mathrm{d}t + x^T(t_1)\Psi x(t_1), \tag{2}$$

where $R \in R^{m\times m}$ and $\Psi \in R^{n\times n}$ are positive definite matrices. This problem can be considered to be a minimum control energy problem with a terminal state penalty.

In [1–3], an optimal control problem with respect to the quadratic cost function, including both the control and state weighting, has been considered. In this case, the resulting optimal control law requires that partial differential equations be solved, which is very difficult. It will be shown that considering the cost function (2) eliminates the need for solving partial differential equations while conserving some optimality of the resulting control law.

The solution of (1) is given by [23]

$$\begin{aligned} x(t) = \Phi(t-t_0)x(t_0) &+ \int_{t_0-h}^{t_0} \Phi(t-s-h)A_1x(s)\mathrm{d}s \\ &+ \int_{t_0}^{t} \Phi(t-s)B_0u(s)\mathrm{d}s \end{aligned} \tag{3}$$

where the state transition matrix $\Phi(t)$ is the solution of the matrix differential-difference equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathrm{t}) = A_0\Phi(t) + A_1\Phi(t-h) \tag{4}$$

with the boundary conditions

$$\Phi(0) = I \text{ and } \Phi(\mathrm{t}) = 0, \quad t < 0.$$

According to (3), the cost function (2) can be rewritten by

$$J = \int_{t_0}^{t_1} u^T(t)Ru(t) + \hat{x}^T(t_1)\Psi\hat{x}(t_1), \tag{5}$$

where

$$\begin{aligned} \hat{x}(t) = &\int_{t_0}^{t} \Phi(t_1-s)B_0u(s)\mathrm{d}s + \Phi(t_1-t_0)x(t_0) \\ &+ \int_{t_0-h}^{t_0} \Phi(t_1-s-h)A_1x(s)\mathrm{d}s, \quad t \in [t_0, t_1]. \end{aligned} \tag{6}$$

It is noted that, by taking $t=t_0$ in (6), $\hat{x}(t)$ can be represented in a closed form

$$\hat{x}(t) = \Phi(t_1-t)x(t) + \int_{t-h}^{t} \Phi(t_1-s-h)A_1x(s)\mathrm{d}s. \tag{7}$$

Note that $\hat{x}(t)$ satisfies

$$\dot{\hat{x}}(t) = \Phi(t_1-t)B_0u(t), \tag{8}$$

which is a differential equation without delay. It is remarkable that an optimization problem for a time-delay system (1) and (2) can be transformed to an optimization problem for a delay-free ordinary system (5)–(8). It is noted that by a reduction transformation method, control-delay systems can be transformed to a delay-free ordinary system [24], which can be used for stabilization. In this sense, the transformation (6) is a reduction transformation method for state-delay systems.

From LQ control theory, the closed-loop optimal control is given by

$$\begin{aligned} u^*(t) = &-R^{-1}B_0^T\Phi^T(t_1-t)P(t) \\ &\times \left[\Phi(t_1-t)x(t) + \int_{t-h}^{t} \Phi(t_1-s-h)A_1x(s)\mathrm{d}s\right], \end{aligned} \tag{9}$$

where $P(t)$ is the solution to the following Ricatti differential equation:

$$\dot{P}(t) = P(t)\Phi(t_1 - t)B_0 R^{-1} B_0 \Phi^T(t_1 - t)P(t) \tag{10}$$

with boundary condition $P(t_1) = \Psi$. It can be shown that $P(t)$ is given by

$$P(t) = \Psi[I + W(t_1 - t)\Psi]^{-1}, \tag{11}$$

where

$$W(t_1 - t) = \int_t^{t_1} \Phi(t_1 - s)B_0 R^{-1} B_0 \Phi^T(t_1 - s)\mathrm{d}s. \tag{12}$$

We apply the receding horizon concept [25,26]. Consider an optimal control of the system (1) which minimizes a moving cost

$$J = \int_t^{t+T} u^T(\tau)Ru(\tau)\mathrm{d}\tau + x^T(t+T)\Psi x(t+T),$$

where T is the horizon length. The receding horizon control can be obtained by replacing t_1 in (9) with $t+T$:

$$\begin{aligned}\hat{u}(t) = &-R^{-1}B_0^T\Phi^T(T)\Psi[I + W(T)\Psi]^{-1}\\ &\times \left[\Phi(T)x(t) + \int_{t-h}^{t} \Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right].\end{aligned} \tag{13}$$

This suggested control law may be the simplest receding horizon control. However, its stability is currently unknown. It is believed the control (13) can stabilize a broad class of time-delay systems because the corresponding control of ordinary systems stabilizes the systems for some large Ψ. It is noted that, unlike for ordinary systems, general stabilizing receding horizon controls for time-delayed system are, as yet, unknown.

Remark 2.1. *The receding horizon control (13) is always defined. Nonsingularity of $[I + W(T)\Psi]$ can be shown as follows:*

$$I + W(T) = \left[\Psi^{-1} + W(T)\right]\Psi.$$

Since $\Psi > 0$ and $\Psi^{-1} + W(T) > 0$, it follows that $[\Psi^{-1} + W(T)]\Psi$ is nonsingular.

Remark 2.2. *The optimal control that minimizes a cost function*

$$J = \int_{t_0}^{t_1} u^T(t)Ru(t)dt \tag{14}$$

subject to a fixed terminal state constraint

$$x(t_1) = 0 \tag{15}$$

can be obtained by requiring $\Psi = \infty I$ in (2). In this case, the receding horizon control is given by

$$\begin{aligned}\hat{u}(t) = &-R^{-1}B_0^T\Phi^T(T)W^{-1}(T)\\ &\times \left[\Phi(T)x(t) + \int_{t-h}^{t} \Phi(t+T-s-h)A_1 x(s)ds\right].\end{aligned} \tag{16}$$

The control (16) is simpler than (13). Note that the receding horizon control is defined only when the controllability matrix (12) is nonsingular, which is equivalent to the condition of pointwise controllability of the time-delay system (1). In general, for time-delay systems, the functionwise controllability is considered as a natural counterpart to the usual controllability of delay-free systems. The receding horizon controller (16) requires only pointwise controllability, which is less restrictive than functionwise controllability.

Remark 2.3. *In the case where the system is not pointwise controllable, we can still apply the proposed control (16) by applying the generalized inverse of the controllability matrix instead of the inverse, $W^{-1}(T)$.*

Remark 2.4. *The values of the state transition matrix $\Phi(T)$ can be obtained by solving (4) numerically [27]. However, in the proposed receding horizon control, we can set $T \leqslant h$ so that we obtain $\Phi(T) = \mathrm{e}^{A_0 T}$, which is simple to compute. The horizon length T is a design parameter in the RHC. It is noted that the state over the horizon is not weighted. Instead the terminal state is weighted. Therefore the shorter horizon length is expected to yield the faster convergence. For the purpose of fast stabilization, taking $T \leqslant h$ is preferred.*

Remark 2.5. *The integral term $\int_{t-h}^{t}\Phi(t+T-s-h)A_1x(s)ds$ in the controller (13) can be realized by a trapezoidal integration method with the step size Δ. It is natural that the smaller Δ we use, the more exact result we get. The selection of Δ depends on the time constant of the system. In general, $\Delta = 0.01$ gives satisfactory result and smaller step size than that yields little difference.*

3. Stability condition for the receding horizon control

It is believed that the suggested control can stabilize many time-delay systems. This will be illustrated by simulations in the next section. However, for rigorous proof of stability, we provide a sufficient LMI condition to check whether the closed-loop system controlled by the receding horizon control (13) or (16) is asymptotically stable or not. The following inequality is introduced [12], which is essential in deriving the sufficient condition provided in this section.

$$2\int_{\Omega} a^T(\alpha)\mathcal{N} b(\alpha)\mathrm{d}\alpha < \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y+\mathcal{N} \\ Y^T+\mathcal{N}^T & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha \tag{17}$$

where

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geqslant 0. \tag{18}$$

$a(\cdot) \in \mathbf{R}^{n_a}$ and $b(\cdot) \in \mathbf{R}^{n_b}$ are defined on the interval Ω and $\mathcal{N} \in \mathbf{R}^{n_a \times n_b}$, $X \in \mathbf{R}^{n_a \times n_a}$, $Y \in \mathbf{R}^{n_a \times n_b}$ and $Z \in \mathbf{R}^{n_b \times n_b}$.

The following theorem gives a sufficient condition for the stability of the proposed receding horizon control.

Theorem 3.1. *If there exist $P, M, S, Q_1, Q_2, X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3, X_4, Y_4, Z_4, X_5, Y_5, Z_5$ and λ such that*

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12}^T & P_{22} & P_{23} & P_{24} \\ P_{13}^T & P_{23}^T & P_{33} & P_{34} \\ P_{14}^T & P_{24}^T & P_{34}^T & P_{44} \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} P & M \\ M^T & S \end{bmatrix} > 0, \quad \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} X_2 & Y_2 \\ Y_2^T & Z_2 \end{bmatrix} \geqslant 0, \tag{20}$$

$$\begin{bmatrix} X_3 & Y_3 \\ Y_3^T & Z_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} X_4 & Y_4 \\ Y_4^T & Z_4 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} X_5 & Y_5 \\ Y_5^T & Z_5 \end{bmatrix} \geqslant 0, \tag{21}$$

$$Z_1 + Z_2 + Z_3 \leq \lambda I \tag{22}$$

where

$$\begin{aligned}
P_{11} &\triangleq A^T P + PA + Q_1 + Q_2 + A^T[(h-T)Z_4 + hZ_5]A \\
&\quad + TX_1 + (h-T)X_4 + hX_5 + Y_4 + Y_4^T + Y_5 + Y_5^T \\
P_{12} &\triangleq MA_1 - Y_4 \\
P_{13} &\triangleq PA_1 - M\Phi(T)A_1 - Y_5 + A^T[(h-T)Z_4 + hZ_5]A_1 \\
P_{14} &\triangleq PB + A^T M + MA_0 + Y_1 + A^T[(h-T)Z_4 + hZ_5]B \\
P_{22} &\triangleq -Q_1 + TX_2 + \lambda A_1^T C A_1 \\
P_{23} &\triangleq 0 \\
P_{24} &\triangleq A_1^T S + Y_2 \\
P_{33} &\triangleq -Q_2 + TX_3 + (h-T)A_1^T Z_4 A_1 + hA_1^T Z_5 A_1 \\
P_{34} &\triangleq A_1^T M - A_1^T \Phi^T(T) S + Y_3 + A_1^T[(h-T)Z_4 + hZ_5]B \\
P_{44} &\triangleq A_0^T S + SA_0 + B^T M + M^T B + B^T[(h-T)Z_4 + hZ_5]B
\end{aligned}$$

$$A \triangleq A_0 - B_0 R^{-1} B_0^T \Phi^T(T)\Psi[I + W(T)\Psi]^{-1}\Phi(T), \tag{23}$$

$$B \triangleq -B_0 R^{-1} B_0^T \Phi^T(T)\Psi[I + W(T)\Psi]^{-1}, \tag{24}$$

$$C \triangleq \int_0^T \Phi^T(\beta)\Phi(\beta)d\beta, \tag{25}$$

then the system (1) with the control (13) for $T \leqslant h$ *is asymptotically stable.*

Proof 1. Choose a Lyapunov functional as

$$V(x(t-\alpha), \dot{x}(t-\alpha), \alpha \in [0,h])$$

$$= V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8$$

where

$$V_1 \triangleq \begin{bmatrix} x(t) \\ \int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1 x(s)\mathrm{d}s \end{bmatrix}^T \begin{bmatrix} P & M \\ M^T & S \end{bmatrix} \times \begin{bmatrix} x(t) \\ \int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1 x(s)\mathrm{d}s \end{bmatrix},$$

$$V_2 \triangleq \int_0^T \int_{t-\beta+T-h}^{t+T-h} x^T(\alpha) A_1^T \Phi^T(\beta)(Z_1+Z_2+Z_3)\Phi(\beta)A_1 x(\alpha)\mathrm{d}\alpha\mathrm{d}\beta,$$

$$V_3 \triangleq \int_0^t \int_{\beta-(h-T)}^{\beta} \begin{bmatrix} x(\beta) \\ \dot{x}(\alpha) \end{bmatrix}^T \begin{bmatrix} X_4 & Y_4 \\ Y_4^T & Z_4 \end{bmatrix} \begin{bmatrix} x(\beta) \\ \dot{x}(\alpha) \end{bmatrix} \mathrm{d}\alpha\mathrm{d}\beta,$$

$$V_4 \triangleq \int_0^t \int_{\beta-h}^{\beta} \begin{bmatrix} x(\beta) \\ \dot{x}(\alpha) \end{bmatrix}^T \begin{bmatrix} X_5 & Y_5 \\ Y_5^T & Z_5 \end{bmatrix} \begin{bmatrix} x(\beta) \\ \dot{x}(\alpha) \end{bmatrix} \mathrm{d}\alpha\mathrm{d}\beta,$$

$$V_5 \triangleq \int_{-(h-T)}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) Z_4 \dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta,$$

$$V_6 \triangleq \int_{-h}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) Z_5 \dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta,$$

$$V_7 \triangleq \int_{t+T-h}^{t} x^T(\alpha) Q_1 x(\alpha)\mathrm{d}\alpha,$$

$$V_8 \triangleq \int_{t-h}^{t} x^T(\alpha) Q_2 x(\alpha)\mathrm{d}\alpha,$$

for

$$\begin{bmatrix} P & M \\ M^T & S \end{bmatrix} > 0, \quad Z_1 + Z_2 + Z_3 \geqslant 0,$$

$$\begin{bmatrix} X_4 & Y_4 \\ Y_4^T & Z_4 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} X_5 & Y_5 \\ Y_5^T & Z_5 \end{bmatrix} \geqslant 0, \quad Q_1 > 0, \quad Q_2 > 0.$$

Since the closed-loop system (1) with the control (13) for $T \leqslant h$ is

$$\dot{x}(t) = \left[A_0 - B_0 R^{-1} B_0^T \Phi^T(T)\Psi(I + W(T)\Psi)^{-1}\Phi(T)\right]x(t) + A_1 x(t-h) - B_0 R^{-1} B_0^T \Phi^T(T)\Psi(I + W(T)\Psi)^{-1} \int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1 x(s)\mathrm{d}s = Ax(t) + A_1 x(t-h) + B\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s, \tag{26}$$

where A and B are as defined in (23) and (24)

From the fact

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s = A_1 x(t+T-h) - \Phi(T)A_1 x(t-h) + A_0\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s,$$

the derivative of V_1 is written as

$$\begin{aligned}\dot{V}_1 &= x^T(t)\left(A^T P + PA\right)x(t) + \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right)^T \\ &\times \left(A_0^T S + SA_0 + B^T M + M^T B\right) \times \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &+ 2x^T(t)(PA_1 - M\Phi(T)A_1)x(t-h) \\ &+ 2x^T(t)(MA_1)x(t+T-h) + 2x^T(t) \times \left(PB + A^T M + MA_0\right) \times \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &+ 2x^T(t+T-h)\left(A_1^T S\right) \times \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &+ 2x^T(t-h)\left(A_1^T M - A_1^T\Phi^T(T)S\right) \times \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right)\end{aligned}$$

Defining $a(\cdot), b(\cdot)$, and $\mathcal{N}$ in (17) as $a(\alpha) \triangleq x(t)$, $b(\alpha) \triangleq \Phi(t+T-\alpha-h)A_1 x(\alpha)$, and $\mathcal{N} \triangleq PB + A^T M + MA_0$ for all $\alpha \in [t-h, t]$ and applying the inequality (17), we have

$$\begin{aligned}&2x^T(t)\left(PB + A^T M + MA_0\right)\left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &\leqslant Tx^T(t) + 2x^T(t)\left(PB + A^T M + MA_0 + Y_1\right) \times \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &+ \int_{t-h}^{t+T-h} x^T(s)A_1^T\Phi^T(t+T-s-h) \times Z_1\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\end{aligned}$$

with the LMI condition

$$\begin{bmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{bmatrix} \geqslant 0.$$

Similarly we have

$$\begin{aligned}&2x^T(t+T-h)\left(A_1^T S\right)\left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &\leqslant Tx^T(t+T-h)X_2 x(t+T-h) + 2x^T\left(A_1^T S + Y_2\right)\left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &+ \int_{t-h}^{t+T-h} x^T(s)A_1^T\Phi^T(t+T-s-h) \times Z_2\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&2x^T(t-h)\left(A_1^T M - A_1^T\Phi^T(T)S\right)\left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &\leqslant Tx^T(t-h)X_3 x(t-h) + 2x^T(t-h)\left(A_1^T M - A_1^T\Phi^T(T)S + Y_3\right) \times \left(\int_{t-h}^{t+T-h}\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\right) \\ &+ \int_{t-h}^{t+T-h} x^T(s)A_1^T\Phi^T(t+T-s-h) \times Z_3\Phi(t+T-s-h)A_1 x(s)\mathrm{d}s\end{aligned}$$

with LMI conditions

$$\begin{bmatrix} X_2 & Y_2 \\ Y_2^T & Z_2 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} X_3 & Y_3 \\ Y_3^T & Z_3 \end{bmatrix} \geqslant 0.$$

Hence, $\dot{V}_1$ satisfies

$$\begin{aligned}
\dot{V}_1 \leqslant\ & x^T(t)\big(A^TP + PA + TX_1\big)x(t) \\
& + Tx^T(t+T-h)X_2x(t+T-h) \\
& + Tx^T(t-h)X_3x(t-h) \\
& + \left(\int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s\right)^T \\
& \times \big(A_0^TS + SA_0 + B^TM + M^TB\big) \\
& \times \left(\int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s\right) \\
& + 2x^T(t)(PA_1 - M\Phi(T)A_1)x(t-h) \\
& + 2x^T(t)(MA_1)x(t+T-h) \\
& + 2x^T(t)\big(PB + A^TM + MA_0 + Y_1\big) \\
& \times \left(\int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s\right) \\
& + 2x^T(t+T-h)\big(A_1^TS + Y_2\big) \\
& \times \left(\int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s\right) \\
& + 2x^T(t-h)\big(A_1^TM - A_1^T\Phi^T(T)S + Y_3\big) \\
& \times \left(\int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s\right) \\
& + \int_{t-h}^{t+T-h} x^T(s)A_1^T\Phi^T(t+T-s-h) \\
& \times (Z_1+Z_2+Z_3)\Phi(t+T-s-h)A_1x(s)\mathrm{d}s.
\end{aligned}$$

The derivative of V_2 is written as

$$\begin{aligned}
\dot{V}_2 =\ & x^T(t+T-h)A_1^T\left[\int_0^T \Phi^T(\beta)(Z_1+Z_2+Z_3)\Phi(\beta)\mathrm{d}\beta\right] \\
& \times A_1x(t+T-h) - \int_{t-h}^{t+T-h} x^T(s)A_1^T\Phi^T(t+T-s-h) \\
& \times (Z_1+Z_2+Z_3)\Phi(t+T-s-h)A_1x(s)\mathrm{d}s
\end{aligned}$$

With the condition (22), $\dot{V}_2$ satisfies

$$\begin{aligned}
\dot{V}_2 \leqslant\ & \lambda x^T(t+T-h)A_1^TCA_1x(t+T-h) \\
& - \int_{t-h}^{t+T-h} x^T(s)A_1^T\Phi^T(t+T-s-h)(Z_1+Z_2+Z_3) \\
& \times \Phi(t+T-s-h)A_1x(s)\mathrm{d}s.
\end{aligned}$$

The derivatives of V_3 to V_8 are represented by

$$\begin{aligned}
\dot{V}_3 =\ & (h-T)x^T(t)X_4x(t) + 2x^T(t)Y_4\int_{t-(h-T)}^{t} \dot{x}(\alpha)\mathrm{d}\alpha \\
& + \int_{t-(h-T)}^{t} \dot{x}^T(\alpha)Z_4\dot{x}(\alpha)\mathrm{d}\alpha \\
\dot{V}_4 =\ & hx^T(t)X_5x(t) + 2x^T(t)Y_5\int_{t-h}^{t} \dot{x}(\alpha)d\alpha \\
& + \int_{t-h}^{t} \dot{x}^T(\alpha)Z_5\dot{x}(\alpha)\mathrm{d}a \\
\dot{V}_5 =\ & (h-T)\dot{x}^T(t)Z_4\dot{x}(t) - \int_{t-(h-T)}^{t} \dot{x}^T(\alpha)Z_4\dot{x}(\alpha)\mathrm{d}\alpha \\
\dot{V}_6 =\ & h\dot{x}^T(t)Z_5\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(\alpha)Z_2\dot{x}(\alpha)\mathrm{d}\alpha \\
\dot{V}_7 =\ & x^T(t)Q_1x(t) - x^T(t+T-h)Q_1x(t+T-h) \\
\dot{V}_8 =\ & x^T(t)Q_2x(t) - x^T(t-h)Q_2x(t-h).
\end{aligned}$$

Using the fact

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + B\int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s$$

$$\int_{t-(h-T)}^{t} \dot{x}(\alpha)\mathrm{d}\alpha = x(t) - x(t+T-h)$$

$$\int_{t-h}^{t} \dot{x}(\alpha)\mathrm{d}\alpha = x(t) - x(t-h)$$

we finally obtain

$$\dot{V} \leqslant \bar{x}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12}^T & P_{22} & P_{23} & P_{24} \\ P_{13}^T & P_{23}^T & P_{33} & P_{34} \\ P_{14}^T & P_{24}^T & P_{34}^T & P_{44} \end{bmatrix} \bar{x} \tag{27}$$

where

$$\bar{x} = \begin{bmatrix} x(t) \\ x(t+T-h) \\ x(t-h) \\ \int_{t-h}^{t+T-h} \Phi(t+T-s-h)A_1x(s)\mathrm{d}s \end{bmatrix}$$

Therefore, if LMIs (19) to (22) are satisfied, $\dot{V}$ will be negative and thus the closed-loop system (26) is asymptotically stable. This completes the proof. □

Since the condition of Theorem 3.1 is formulated in terms of LMIs, it can be easily checked using convex optimization algorithms [21]. It is noted that even if the above sufficient condition is not met, the receding horizon control (13) can still stabilize the systems. In the next section, we will provide an algorithm, as a necessary and sufficient condition, to check the closed-loop stability by calculating the eigenvalues of time-delay systems directly.

Remark 3.1. *The stability of the system (1) with the control (16) can also be checked by Theorem 3.1. In this case, A and B are given differently from those in (23) and (24) as follows:*

$$A \triangleq A_0 - B_0 R^{-1} B_0^T \Phi^T(T) W^{-1}(T)\Phi(T),$$

$$B \triangleq -B_0 R^{-1} B_0^T \Phi^T(T) W^{-1}(T).$$

4. Eigenvalue searching algorithm for distributed delay systems

The state delayed system (1) with the proposed RHC (13) leads to the closed-loop system

$$\begin{aligned}\dot{x}(t) = {} & \left[A_0 - B_0 R^{-1} B_0^T \Phi^T(T)\Psi(I + W(T)\Psi)^{-1}\Phi(T)\right]x(t) \\ & + A_1 x(t-h) - B_0 R^{-1} B_0^T \Phi^T(T)\Psi(I + W(T)\Psi)^{-1} \\ & \times \int_{t-h}^{t} \Phi(t + T - s - h) A_1 x(s)\mathrm{d}s.\end{aligned} \tag{28}$$

To check the necessary and sufficient condition for the closed-loop stability, we have to find if there is any eigenvalue of (28) whose real part is positive or equal to zero. For this purpose, we will provide a numerical algorithm to compute the eigenvalues of general distributed-delay systems.

Let us consider a system

$$\begin{aligned}\dot{x}(t) = {} & K_0 x(t) + \sum_{l=1}^{N_x} K_l x\left(t - h_{x_l}\right) \\ & + \sum_{l=1}^{N_d} \int_{-h_{d_l}}^{0} D_l(\theta) x(t+\theta)\mathrm{d}\theta, \qquad t \geqslant 0\end{aligned} \tag{29}$$

where K_0, $K_l \in \mathcal{R}^{n\times n}$, h_{x_l} and h_{d_l} are positive delay parameters, and elements of $D_l(\theta)$ have bounded variations on $\left[-\max\left(h_{d_l}\right), 0\right]$. It can be seen that (28) is a special case of (29). The characteristic function corresponding to (29) is given by

$$\Delta(\lambda) = \det\left(\lambda I - K_0 - \sum_{l=1}^{N_x} \mathrm{e}^{-h_{x_l}\lambda} K_l - \sum_{l=1}^{N_d} \int_{-h_{d_l}}^{0} \mathrm{e}^{\lambda\theta} D_l(\theta)\mathrm{d}\theta\right). \tag{30}$$

It is well known that the trivial solution $x(t) = 0$ of the time-delay system (29) is stable if and only if the characteristic function (30) has no zeros such that *Re* $\lambda \geqslant 0$ and the kernel $D_l(\theta)$ have bounded variation [28].

Consider the Taylor series for the exponential terms of (30) around a certain point λ_0:

$$e^{-h_{x_l}\lambda} = \mathrm{e}^{-h_{x_l}\lambda_0} \sum_{k=0}^{\infty} \frac{\left(-h_{x_l}\right)^k (\lambda - \lambda_0)^k}{k!}, \tag{31}$$

$$e^{\lambda\theta} = \mathrm{e}^{\lambda_0\theta} \sum_{k=0}^{\infty} \frac{\theta^k (\lambda - \lambda_0)^k}{k!}. \tag{32}$$

Since it is impossible to calculate the summation up to infinity, we truncate (31) and (32) at N terms, such that

$$\mathrm{e}^{-h_{x_l}\lambda_0} \sum_{k=0}^{N-1} \frac{\left(-h_{x_l}\right)^k (\lambda - \lambda_0)^k}{k!}, \tag{33}$$

$$\mathrm{e}^{\lambda\theta} \simeq \mathrm{e}^{\lambda_0\theta} \sum_{k=0}^{N-1} \frac{\theta^k (\lambda - \lambda_0)^k}{k!}. \tag{34}$$

Then (30) reduces to an *approximate* characteristic function

$$\hat{\Delta}(\lambda) = \det\left(\sum_{k=0}^{N-1} (\lambda - \lambda_0)^k G_k\right) \tag{35}$$

where

$$G_0 = \lambda_0 I - K_0 - \sum_{l=1}^{N_x} \mathrm{e}^{-h_{x_l}\lambda_0} K_l - \sum_{l=1}^{N_d} \int_{-h_{d_l}}^{0} \mathrm{e}^{\lambda_0\theta} D_l(\theta)\mathrm{d}\theta, \tag{36}$$

$$G_1 = I + \sum_{l=1}^{N_x} h_{x_l} \mathrm{e}^{-h_{x_l}\lambda_0} K_l - \sum_{l=1}^{N_d} \int_{-h_{d_l}}^{0} \mathrm{e}^{\lambda_0\theta} \theta K_2(\theta)\mathrm{d}\theta, \tag{37}$$

$$\begin{aligned}G_k = {} & -\frac{1}{k!} \sum_{l=1}^{N_x} \mathrm{e}^{-h_{x_l}\lambda_0} \left(-h_{x_l}\right)^k K_l \\ & - \frac{1}{k!} \sum_{l=1}^{N_d} \int_{-h_{d_l}}^{0} \mathrm{e}^{\lambda_0\theta} \theta^k D_l(\theta)\mathrm{d}\theta, \quad 2 \leqslant k \leqslant N-1.\end{aligned} \tag{38}$$

Now, we have the following theorem.

Theorem 4.1. *Let* $\widehat{\lambda_j}$ *be the* j*th zero of the approximate characteristic function (35). Then* $\widehat{\lambda_j}$ *is given by*

$$\widehat{\lambda_j} = x_j + \lambda_0, \quad j = 1, 2, \ldots, n(N-1) \tag{39}$$

where x_j is the jth root of

$$\sum_{m=0}^{n(N-1)}\left(\sum_{\Omega}\det\begin{bmatrix}\bar{g}_1^{i_1}\\ \bar{g}_2^{i_2}\\ \vdots\\ \bar{g}_n^{i_n}\end{bmatrix}\right)x^m=0 \tag{40}$$

in which Ω is the set of all possible permutations of the integers $(i_1, i_2,\ldots,i_n)$ satisfying

$$i_1+i_2+\cdots+i_n=m,\quad 0\leqslant i_1,i_2,\ldots,i_n\leqslant N-1$$

and $\bar{g}_k^{i_k}$ is the k-th row of G_k defined by (36)–(38).

Proof. Consider

$$\hat{\Delta}_0(x)=\det\left(\sum_{k=0}^{N-1}x^kG_k\right)=0 \tag{41}$$

where G_k is defined by (36)–(38). Next, we define $a_{ij}(x)$ as the (i,j) th element of $\sum_{k=0}^{N-1}x^kG_k$ such that

$$a_{ij}(x)=g_{ij}^{N-1}x^{N-1}+g_{ij}^{N-2}x^{N-2}+\cdots+g_{ij}^1x+g_{ij}^0 \tag{42}$$

where g_{ij}^k is the (i,j)th element of G_k. From the well known property of determinants [29], it follows that

$$\hat{\Delta}_0(x)=\det([a_{ij}(x)])=\sum_{\kappa}(\pm)a_{1k_1}(x)a_{2k_2}(x)\cdots a_{nk_n}(x) \tag{43}$$

where κ is the set of all possible permutations of the integers from 1 to n, and the $(-)$ sign appears if the number of transpositions in the permutation is odd. Now, from (42), it is seen that

$$\begin{aligned}a_{1k_1}(x)a_{2k_2}(x)\cdots a_{nk_n}(x)\\ &=\left(g_{1k_1}^{N-1}x^{N-1}+g_{1k_1}^{N-2}x^{N-2}+\cdots+g_{1k_1}^1x+g_{1k_1}^0\right)\\ &\quad\left(g_{2k_2}^{N-1}x^{N-1}+g_{2k_2}^{N-2}x^{N-2}+\cdots+g_{2k_2}^1x+g_{2k_2}^0\right)\\ &\quad\cdots\left(g_{nk_n}^{N-1}x^{N-1}+g_{nk_n}^{N-2}x^{N-2}+\cdots+g_{nk_n}^1x+g_{nk_n}^0\right)\\ &=\sum_{m=0}^{n(N-1)}\left(\sum_{\Omega}g_{1k_1}^{i_1}g_{2k_2}^{i_2}\cdots g_{nk_n}^{i_n}\right)x^m\end{aligned} \tag{44}$$

where Ω is as defined in Theorem 4.1. Hence, it follows from (43) and (44) that

$$\begin{aligned}\hat{\Delta}_0(x)&=\sum_{\kappa}(\pm)\sum_{m=0}^{n(N-1)}\left(\sum_{\Omega}g_{1k_1}^{i_1}g_{2k_2}^{i_2}\cdots g_{nk_n}^{i_n}\right)x^m\\ &=\sum_{m=0}^{n(N-1)}\sum_{\Omega}\left(\sum_{\kappa}(\pm)g_{1k_1}^{i_1}g_{2k_2}^{i_2}\cdots g_{nk_n}^{i_n}\right)x^m\\ &=\sum_{m=0}^{n(N-1)}\left(\sum_{\Omega}\det\begin{bmatrix}\bar{g}_1^{i_1}\\ \bar{g}_2^{i_2}\\ \vdots\\ \bar{g}_n^{i_n}\end{bmatrix}\right)x^m\end{aligned}$$

where $\bar{g}_k^{i_k}$ is the k th row of G_{i_k}.

If x_j is the jth root of (41), it is obvious from (35) and (41) that $\lambda_j=x_j+\lambda_0$ is the corresponding root of the approximate characteristic equation, $\bar{\Delta}(\lambda=0)$. This completes the proof. □

Using Theorem 4.1, we can transform the approximate characteristic equation $\bar{\Delta}(\lambda=0)$ into a polynomial equation which can be solved using well-established numerical methods. However, it should be noted that the roots obtained are only valid in a small region around the center point of the Taylor expansion, λ_0. We propose an eigenvalue searching algorithm using Theorem 4.1. Some useful properties of time-delay systems will be exploited here, such as

(1) An upper bound on the magnitude of Right Half Plane (RHP) eigenvalues can be computed [22].
(2) The number of eigenvalues inside a given disk centered at the origin can be computed [30].
(3) Eigenvalues are asymptotically distributed on a finite number of chains [28].

First, we note that it is impossible to find all eigenvalues of time-delay systems since they have an infinite number of eigenvalues. However, to check the stability of a system, we do not need to compute all the eigenvalues on the whole complex plane. Instead, we only need to check for the existence of eigenvalues on the RHP. Fortunately, it is known that time-delay systems have a finite number of eigenvalues on the RHP. Moreover, an upper bound on the magnitude of RHP eigenvalues can be computed as follows [22]:

$$|\lambda|\leqslant\|K_0\|+\sum_{l=1}^{N_x}\|K_l\|+\sum_{l=1}^{N_d}\int_{-h_{d_l}}^{0}\left|D_l(\theta)\right|\mathrm{d}\theta\quad\text{for }\mathrm{Re}\lambda\geqslant 0. \tag{45}$$

Therefore, setting the above upper bound as the radius, we consider a disk centered at the origin. Since it is guaranteed that there exist no RHP eigenvalues outside this disk, we search eigenvalues only inside this disk region.

Next, we use the result of [3] which enables us to know how many eigenvalues exist inside this disk in advance. Using this pre-computed information, we can see if we have found all existing eigenvalues to stop the iterations.

Finally, we make use of the observation that the eigenvalues of time-delay systems are asymptotically distributed on a finite number of chains [28]. This means that the eigenvalues of time-delay systems are not scattered arbitrarily, but are located around one such chain. Therefore, choosing the center point λ_0 in Theorem 4.1 as the eigenvalue found in the previous step, we can find the eigenvalues very efficiently.

Combining these ideas, an eigenvalue searching algorithm can be devised as follows.

4.1. *Eigenvalue searching algorithm 4.1 (main algorithm)*

(1) Compute the upper bound (45) on the magnitude of RHP eigenvalues, which gives the radius of the disk in which the search will be performed.
(2) Compute the total number of eigenvalues inside the search disk using the method in [30].
(3) Set the initial λ_0 as the origin.
(4) Solve (40) to obtain $n(N-1)$ approximate roots $\hat{\lambda}$ satisfying (39).
(5) Refine each root using the following procedure, and select valid eigenvalues among them.
    5.1 Among $n(N-1)$ roots found in Step 4 of Algorithm 4.1, select the refinement candidates $\bar{\lambda}$ with the roots satisfying $|\Delta(\hat{\lambda})| < \varepsilon_0$, and double the polynomial order N previously used.
    5.2 For each candidate $\bar{\lambda}$, set $\lambda_0 = \bar{\lambda}$ and run the following loops. a. Find $n(N-1)$ approximate roots (39) around λ_0 by solving (40). Among the roots found, define $\tilde{\lambda}$ as the root that is nearest to λ_0. b. If $|\Delta(\tilde{\lambda})| < \varepsilon_f$, accept $\tilde{\lambda}$ as a valid eigenvalue, and select the next candidate $\bar{\lambda}$. Else if $|\Delta(\tilde{\lambda})| > |\Delta(\lambda_0)|$, double the polynomial order N and go to Step 2a. Else, sst $\lambda_0 = \tilde{\lambda}$ and go to Step 2a.
(6) If an eigenvalue outside the disk is found, go to Step 7.
    Else if no new eigenvalue is found, double the polynomial order N and go to Step 4.
    Else, set λ_0 as the newly found eigenvalue with the largest magnitude and go to Step 8.
(7) If the total number of eigenvalues found is equal to the number pre-computed in Step 2, then quit. Else, choose a different λ_0 among the eigenvalues found in the first loop (i.e., the eigenvalues found when λ_0 was the origin) and go back to Step 4.

The refinement procedure in Step 5 is needed because the approximations (33) and (34) are valid only in a small neighborhood of λ_0. Through the procedure, more accurate eigenvalues can be obtained than the values found in the first step. In the algorithm, ε_0 and ε_f are tolerance parameters to be set by the user.

We will give examples in the next section to verify that the proposed algorithm is useful for checking the closed-loop stability of the RHC suggested in this paper.

5. Examples

In this section, two numerical examples are presented to illustrate the proposed methods. The first example is an open-loop stable time-delay system for a chemical reactor. The second one is an open-loop unstable liquid monopropellant rocket motor.

Example 5.1. Consider an example of a typical control problem occurring in the chemical and petroleum industries [4]. The block diagram in Fig. 1 shows a refining plant.

Raw materials A and B enter the chemical reactor and take part in three chemical reactions that produce a product P along with some other byproducts. F_A and F_B represent the feed rates (in pounds per hour) of raw

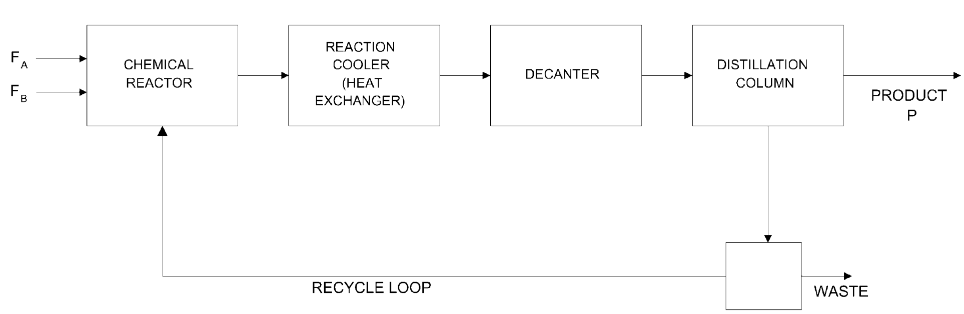

Fig. 1. Refining plant.

materials A and B, respectively. The linearized (time-scaled) equations for the chemical reactor are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -4.93a(t) + 1.92a(t-1) - 1.01b(t) + \frac{\delta F_{\mathrm{A}}}{6V_{\mathrm{R}}}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -3.20a(t) - 5.03b(t) + 1.92b(t-1) - 12.8c(t) + \frac{\delta F_{\mathrm{B}}}{6V_{\mathrm{R}}}$$

$$\frac{\mathrm{d}c}{\mathrm{d}t} = 6.40a(t) + 0.347b(t) - 32.5c(t) + 1.87c(t-1) - 1.04p(t),$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0.833b(t) + 11.0c(t) - 3.96p(t) + 0.724p(t-1).$$

Here, one time unit is 10 min, δF_A is the deviation from the nominal value of the feed rate of material A in pounds per hour, V_R is the pound-volume of the chemical reactor, δF_B is the deviation of the feed rate of material B, $a(t)$ is the deviation in the weight composition of reactant A from its nominal value, $b(t)$ is the deviation in the weight composition of reactant B, $c(t)$ is the deviation in the weight composition of an intermediate product C, and $p(t)$ is the deviation in the weight composition of the product P. Letting $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = p$, $u_1 = \frac{\delta F_{\mathrm{A}}}{6\ V_{\mathrm{R}}}$, and $u_2 = \frac{\delta F_{\mathrm{B}}}{6\ V_{\mathrm{R}}}$, we can write the system

$$\dot{x} = A_0 x(t) + A_1 x(t-1) + Bu(t) \tag{46}$$

where

$$A_0 = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The weight matrix R and the terminal state weight matrix Ψ are chosen as follows:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi = 10\,000 \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$$

For $T = 0.2$, 0.6, and 1, the closed-stability was checked using LMI conditions in Theorem 3.1. It turned out that LMIs are all feasible, which means that the closed-loop system is asymptotically stable. In case of $T = 0.6$, the receding horizon control is given by

$$u(t) = \begin{bmatrix} -0.6851 & -0.1330 & -0.2732 & -0.8547 \\ -0.1330 & -0.2103 & -0.1234 & -0.5562 \end{bmatrix} x(t) + \begin{bmatrix} -0.859 & 1.3995 & 14.7132 & -10.8215 \\ -0.6695 & -0.4512 & 10.3984 & -6.6652 \end{bmatrix} \int_{t-1}^{t-0.4} \mathrm{e}^{A_0(t-s-0.4)} A_1 x(s) \mathrm{d}s. \tag{47}$$

The receding horizon controllers obtained for $T = 0.2$, 0.6 and 1 have been applied to the system with initial state $\phi_1(\theta) = 0.1$, $\phi_2(\theta) = \phi_3(\theta) = \phi_4(\theta) = 0$, $-1 \leqslant \theta \leqslant 0$. Fig. 2 compares the state trajectories x_1 resulting from the proposed controller with the one resulting from the method in [4]. It is seen that the uncontrolled system is very sluggish. It is well illustrated in Fig. 2 that the receding horizon controller with the shorter horizon length yields the faster response. In case of $T = 0.2$, the proposed controller outperforms the one proposed in [4].

To illustrate the sensitivity of the proposed RHC against the variation in the delay size, we designed a receding horizon controller assuming $T = 0.6$ and $h = 1$ and applied it to the system with different delay sizes. Fig. 3 compares the state trajectories for x_1, from which we can say that the proposed RHC is robust against the variation in the delay size.

Example 5.2. Consider a liquid monopropellant rocket motor with a pressure feeding system in [31]. A linearized version of the feeding system and combustion chamber equations produces the state-space model (1) with $h = 1$ and

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

For this example, we designed a controller with a terminal state constraint, that is, $x(t+T) = 0$. The weight matrix R is chosen as $R = 1$. Since this system is not pointwise controllable, but pointwise stabilizable, we could design a controller (16) using the generalized inverse of the controllability matrix (12). The closed-loop stability was tested using the LMI conditions of Theorem 3.1 for $T = 0.6$ and 1. It turned the out that the LMI conditions are feasible, which means the closed-loop system is asymptotically stable.

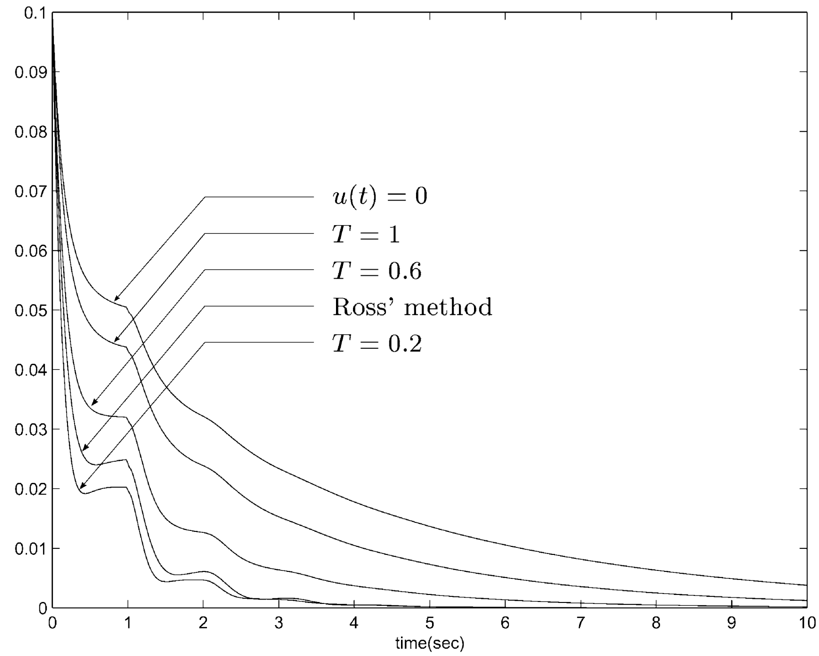

Fig. 2. State x_1: chemical reactor model.

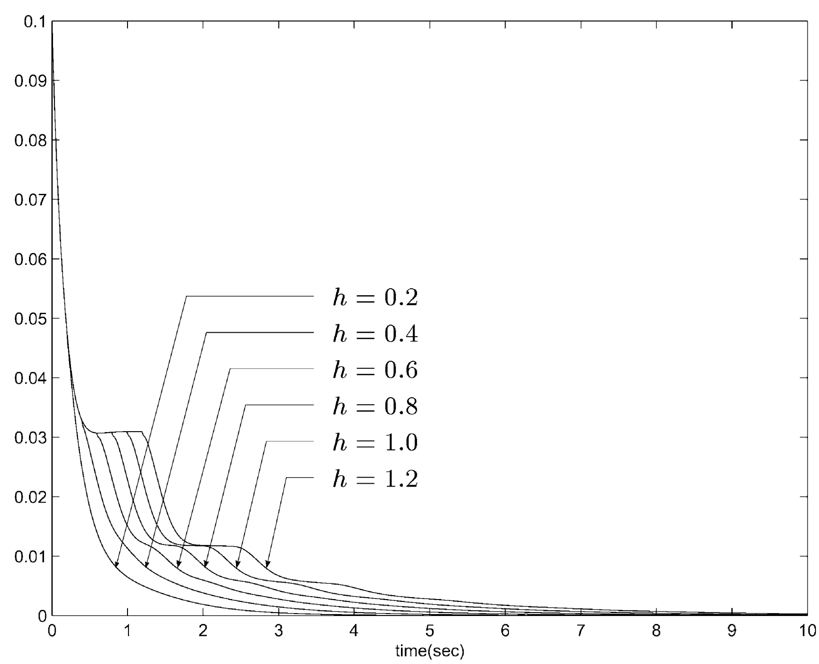

Fig. 3. State x_1 for different delay sizes: chemical reactor model.

Fig. 4 compares the state trajectories x_1 resulting from the proposed controller with the one resulting from the method in [31] in case that the initial condition is $\phi_1(\theta) = \phi_2(\theta) = \phi_2(\theta) = \phi_4(\theta) = 1, \quad -1 \leqslant \theta \leqslant 0.$ The controller proposed in [31] is given by

$$u(t) = -K_c\left[x(t) + \int_{t-1}^{t} e^{A_c(t-h-s)} A_1 x(s) ds\right],$$

where

$$K_c = \begin{bmatrix} -7.6801 & 3.8526 & -2.8497 & 3.5650 \end{bmatrix},$$

$$A_c = \begin{bmatrix} -1.0193 & 0 & 1.0193 & 0 \\ -0.4986 & 0 & 0.4986 & -1 \\ -2.8718 & 0 & 0.8718 & 1 \\ -0.9215 & 1 & -0.0785 & 0 \end{bmatrix}.$$

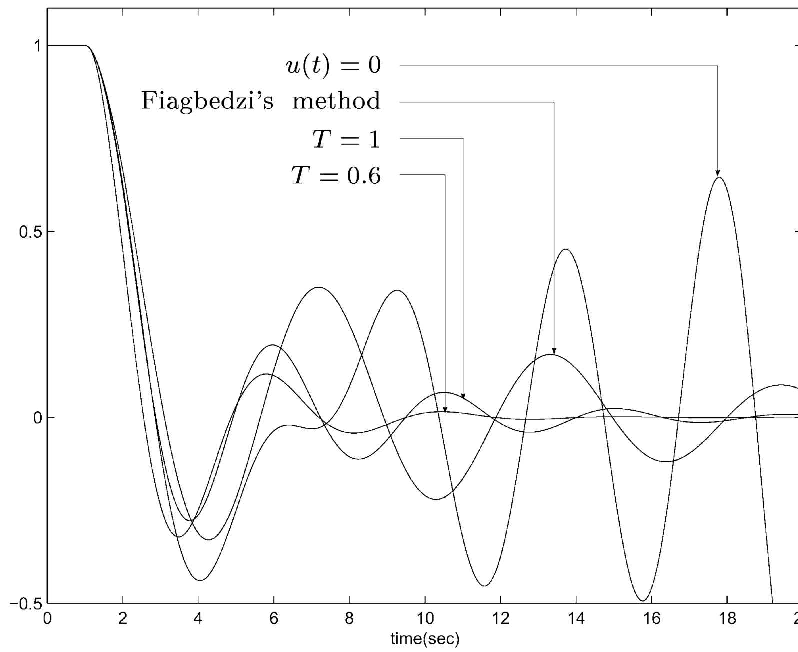

Fig. 4. State x_1: liquid propellant rocket model.

Fig. 4 shows that the uncontrolled system is unstable. The receding horizon controllers obtained for both $T=0.6$ and $T=1$ stabilize the system faster than the the controller of [31].

In order to clearly illustrate the stability, we applied the proposed eigenvalue searching algorithm of Algorithm 4.1 for $T=1$. The eigenvalues of the open-loop systems are $-0.1862\pm0.9179i$, $0.1125\pm1.5201i$, and -1.9745. This shows that the open-loop system is unstable. The eigenvalues of the closed-loop system are $-0.5076\pm0.9159i$, $-2.6094\pm3.0678i$, $-2.0555\pm7.4449i$, and $-2.6542\pm13.8761i$. We see that all unstable poles have been moved to the stable region. Fig. 4 illustrates the closed-loop state trajectories, which clearly show the stabilizing effect.

6. Conclusions

This paper proposes a simple receding horizon control for state delayed systems, while steady state stabilizing LQ regulating controls for time-delay systems are usually very complex. The proposed controller is very simple to construct and thus easily implemented in real applications. To check the closed-loop stability of the proposed controller, a sufficient condition in terms of linear matrix inequalities is proposed. Also, a numerical algorithm is presented for computing the eigenvalues of general distributed delay systems, which can be used for the necessary and sufficient stability check of the proposed controller. Although guaranteed stability of the suggested control is not known, it is believed that this control can stabilize time-delay systems that can be stabilized by other control methods. Since the proposed control is a simple and easy control, we suggest use of this control as the first candidate for stabilizing control for the delay systems. However, in this case, its stability must be checked, possibly by the methods suggested in this paper.

Acknowledgements

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